ON UNIFIED THEORIES OF THERMAL AND SHOT NOISE*

By S. DEB

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ABSTRACT. The paper presents an outline of the principal characteristics of the thermal and the shot types of noise (in circuits and in valves respectively) and examines the similar features of the two phenomena The unified theories of the two types of noises proposed recently by Campbell and Francis, and by Fürth are reviewed and discussed critically. The phenomenon of space-charge reduction of noise has been explained from a new statistical standpoint. It is shewn that although the two types of noises owe their origin to the same basic phenomenon, they differ in one essential respect, *viz*: the presence of thermal equilibrium in the one (thermal) and its absence in the other (shot) It is concluded that thermal noise can be looked upon as a special case of shot noise when thermal equilibrium exists As such, the unified thermo-dynamical theory of Furth becomes unacceptable.

INTRODUCTION

The possibility of existence of spontaneous fluctuations of current in a conductor was first suggested by Einstein (1906). Einstein showed that the mean-square-charge crossing any cross-section of a conductor (of resistance R) in thermal equilibrium, over a finite time interval τ is given by

$$\overline{Q^2}=\frac{2kT}{R}\tau \qquad (1.1)$$

Nyquist (1928) showed that the mean-square fluctuation voltage appearing at the output of a network over the entire frequency range would be

$$^2=\frac{2kT}{\pi R}\int^{\infty}|Z(\omega)|^2 d\omega \qquad (1.2)$$

provided classical law of equipartition holds. In (1.2), $Z(\omega)$ is the transfer impedence of the circuit over the frequency range f to $f+df$. If $Z(\omega)=R$ then over the range f to $f+df$

$$\overline{de^2}=4RkTdf \qquad \ldots \quad (1.3)$$

The possibility of the existence of a similar type of fluctuations of current as obtained in a valve was suggested by Schottky (1918). Schottky gave

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the following expression for the mean-square-fluctuation current over the frequency range f to $f+df$ associated with a diode valve carrying a current i_s under temperature-limited condition,

$$\overline{di^2}=2\epsilon i_s df \qquad \ldots \quad (1.4)$$

where ϵ is the electronic charge. (This relation holds for retarding fields as well. For space-charge-limited condition (1.4) has to be multiplied by a factor Γ^2 which is much less than unity).

From the very beginning, thermal noise and shot noise have been looked upon as two distinct phenomena owing their origin to entirely different physical causes. The former was assumed to be due to the thermal agitational motion of electrons and the latter to the granular nature of electrical carriers, *viz.* the electrons. Only recently it has been recognised that the two types of noise are only two different aspects of the same basic phenomenon. This is understood if one remembers that the existence of random thermal agitation of electrons in a resistance is only possible because the electrons are granular. Also, electrons emitted by heating are the thermal electrons and not the Fermi electrons. As such, any observed fluctuation in emission is entirely due to this fact. Starting with these premises, attempts have been made in recent years to develop a unified theory which embraces both the types of noises. Campbell and Francis (1946) have given such a theory based on statistical reasonings. Fürth (1948) has given another based on thermodynamic reasonings. It is the purpose of this paper to critically review these two theories and to show that a better approach to the problem is made if thermal noise is looked upon as a special case of shot noise as encountered in a valve.

SIMILARITIES IN SHOT AND THERMAL NOISE

The first suggestion of a possible identity of shot and thermal noise arose out of the phenomenon of space-charge-reduction of noise. It was argued that a valve carrying space-charge-limited current differs from one carrying a temperature-limited current only in having a differential resistance R_a. The reduction effect was, therefore, sought to be explained in terms of thermal noise in R_a. But this attempt was unsuccessful. A striking correspondence was, however, established by Williams (1936) under retarding field condition. The expression for valve current under retarding field condition is given by

$$i=i_s \exp\left[-\frac{\epsilon V}{kT_c}\right]$$

and $$R_a = \left(\frac{dV}{di}\right)_{i_s} = \frac{i\epsilon}{kT_c}$$

Hence,
$$\overline{di^2} = 2i\epsilon df = 2R_a kT_c df$$
$$= 4R_a k(T_c/2)df \qquad \ldots \quad (2.1)$$

Relation (2.1) shows that a valve, under condition of retarding field, behaves as an ordinary resistance R_a at half the cathode temperature. North (1940) has given a more general expression for valve noise. According to him the mean-square-fluctuation-current in a valve is always expressible in the form,

$$\overline{di^2} = 4R_a kT_c \theta df \qquad \ldots \quad (2.2)$$

where, θ has a value $\frac{1}{2}$ under retarding field condition and approximately 0.644 under space-charge-limited condition. The value of θ changes rapidly, increasing without limit, as the condition of saturation is approached. North was also able to show that Nyquist's formula (1.2) holds for a valve in thermal equilibrium.

UNIFIED THEORIES: CAMPBELL AND FRANCIS

(*a*) *Campbell and Francis's treatment* :

The method is based upon two theorems, the 'mean' and the 'mean-square' theorems which are collectively known as Campbell's theorems'. These theorems are helpful in calculating the effect of random-fluctuation-noise in any electrical device having a linear response characteristic and may be explained as follows :

Let us observe any random process-thermionic emission, for example, and divide the time preceding the instant of observation into a large number of intervals τ. Suppose a is the average number of particles emitted per unit time. Then $\nu = a\tau$ may be looked upon as the probability of emission of a particular particle within a selected time interval. Suppose now that we observe the effect of these random events in a device whose response is linear. If the instantaneous effects observed in the device due to all the random events occurring within a time t prior to the instant of observation be $S(t)$, then Campbell's theorems state : *In the limit when $\tau \to 0$ and Poisson's distribution law for the events of low probabilities holds, the mean- effect observed in the network will be given by*

$$\bar{y} = a\int_0^\infty S(t)\,dt \qquad \ldots \quad (3.1)$$

and the mean-square-effect by

$$\overline{(y-\bar{y})^2} = a\int_0^\infty [S(t)]^2 dt = aS \text{ (say)} \qquad \ldots \quad (3.2)$$

where S is the value of the integral on the left hand side.

The value of $[S(t)]^2$ can be calculated by the method of Fourier integrals. Campbell and Francis have shewn that if the elementary events giving rise to the fluctuations are a series of current impulses of strength $X_0\tau$, then

$$\overline{(y-\bar{y})^2} = \frac{a}{\pi}\int_0 X_0^2\tau^2[\phi(j\omega)]^2 d\omega \quad \cdots \quad (3.3)$$

where X_0 = height of the impulses,

τ = duration of the impulses

and $\phi(j\omega)$ = frequency response of the network in which the noise is being observed.

Eq. (3.3) can now be applied to deduce expressions for the mean-square-fluctuation-noise, of both the types—shot noise in valves (Schottky's equation) and thermal noise in resistive circuits (Nyquist's equation).

(*b*) *Shot noise*

For the shot noise three cases may be distinguished :

(*i*) When the valve carries temperature-limited current, (ii) when it carries space-charge-limited current and (iii) when the valve is under a retarding field.

Of these, Campbell and Francis have discussed fully only the first one. The second and the third cases were not fully discussed by them because their theorems are not applicable to non-linear device. We give below discussions of all the three cases. The first as developed by Campbell and Francis and second and third as developed by the author of this paper.

Case (*i*). *Valve carrying temperature-limited current.*—Schottky's equation for noise in temperature-limited current in a valve can be easily derived from (3.3) by the general method suggested by Campbell and Francis.

Let us consider the case when the noise is observed directly in the plate circuit of the valve. If the mean square-fluctuation in current is observed then

$$[\phi(j\omega)] = |Y(\omega)| = 1 \quad \cdots \quad (3.4)$$

where $Y(\omega)$ = transfer admittance of the network over ω to $\omega+d\omega$.

The elementary events giving rise to shot noise is the passage of a charge ϵ over a small interval of time τ during which a small current i_τ flows. Thus

$$X_0 = i_\tau \text{ and } X_0\tau = \epsilon \quad \cdots \quad (3.5)$$

and from (3.3), (3.4) and (3.5)

$$\overline{(y-\bar{y})^2} = \overline{i^2} = \frac{a\epsilon^2}{\pi}\int_0 d\omega \quad \cdots \quad (3.6)$$

If observations are restricted over a frequency range f to $f+df$, then because $a\epsilon$ = average rate of flow of charge

$$\overline{di^2} = 2a\epsilon^2\, df = 2i_s \epsilon\, df$$

This is identical with the expression given by Schottky.

Case (ii). Valve carrying space-charge-limited current.—This case has not been fully discussed by Campbell and Francis as their equation holds only for linear device and the caracteristic of a valve under space-charge-limited condition is not linear. However, if one takes into consideration the portion of the curve which is linear, one may arrive at an expression for shot noise as reduced by space-charge effects.

We note that fluctuation voltages are usually very small and for such small ranges of values the characteristics may be taken approximately as linear. Further, under many practical conditions of operation of diode and triode valves reasonably accurate results are obtained by assuming them to behave as linear device and in these cases too, space-charge-reduced noise equation holds. It is likely that in these cases at least individual events separated by infinitesimal time interval would add up linearly. We can therefore, treat the valve an approximately linear device and try to get some qualitative interpretation of space-charge-reduction of noise. In figure 1, has

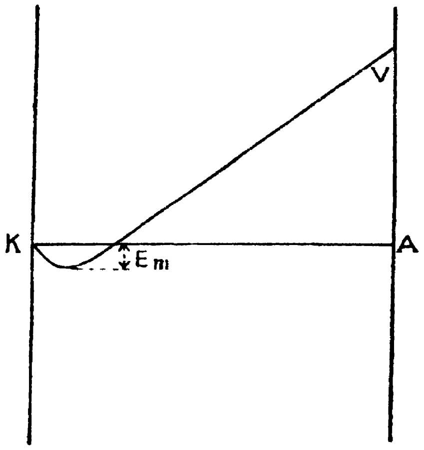
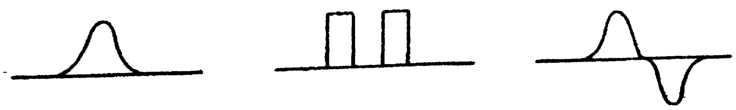


FIG. 1

been shown a voltage distribution curve for a valve carrying space-charge-limited current. Here the passage of an electron from cathode to anode can be considered under two parts :

(I) A part in which the journey is performed in a retarding field between the cathode and the negative potential dip E_m. Due to this the current in the plate circuit will be as shewn in figure. 2 (*a*).



(*a*) (*b*) (*c*)

FIG. 2

(II) A part in which the journey is performed in an accelerating field between E_m and the anode. The corresponding current in the circuit will also be of the form shewn in figure. 2 (*a*). The combination of these two current component can be looked upon as a pair of impulses as shewn in figure. 2 (*b*). Since each of these pulses depicts the transit of an electron over only a part of the distance between the cathode and the anode, the strength of these impulses $X_0\tau$ will now be less than ϵ. If we write it as $\Gamma\epsilon$ ($F << 1$), then it follows from (3.3), (3.5) and (3.6) that

$$\overline{di^2} = 2\epsilon i\Gamma^2 df \; (\Gamma^2 << 1)$$

As regards those electrons which fail to cross the potential dip and return to the cathode it may be observed that they give rise to doublet impulses of the type shown in figure, 2 (*c*). It may be shewn that so long as the frequencies are low enough to make the transit angle small, contribution due to these impulses will be negligible.

If the two pulses are of equal strength then $\Gamma^2 \simeq 25$ which is of the order of the value as obtained experimentally. Value of Γ would depend on the potential distribution, geometry of the valve and the energy distribution of the electrons.

It should be noted that a clear resolution between the two pulses, as shewn in figure 2 (*b*), is possible only under condition of complete space-charge-limitation. In the general case there may be overlapping of the two to some extent.

(*iii*) *Valve under retarding field condition.*—This case has also not been considered by Campbell and Francis on account of lack of linearity. However, the above considerations may also be applied to provide a simple explanation of the nature of noise equation for a valve under retarding field. For this case the potential decreases monotonously from cathode to anode. The passage of an electron to the anode would, therefore, be represented by an impulse of the type shewn in figure 2 (*a*), but having a much greater strength. Since this pulse depicts the passage of one electron from cathode to anode the strength would again be ϵ. Thus it is obvious that Schottky's relation should hold for this region.

(*b*) *Thermal Noise.*

Nyquist's equation (eqn. 1.2) for thermal noise has also been deduced by Campbell and Francis from (3.3). The method of deduction, however, does not make it evident that the condition of thermal equilibrium is essential. The following slightly modified deduction by the author is believed to make this clear in a simple manner.

Let us imagine that the source of thermal noise is composed of a large number of constant current generators in series such that the *k* th generator gives rise to a current pulse i_k lasting over a time τ_k and that these are uncorrelated with respect to time. Thus

$$S'_k(j\omega)=i_k\tau_k \qquad \dots \quad (3.7)$$

and

$$S_k=\frac{1}{\pi}i_k^2\tau_k^2\int_0^\infty[\phi(j\omega)]^2d\omega \qquad \dots \quad (3.8)$$

From (3.2)

$$\overline{(y-\bar{y})^2}=\frac{1}{\pi}\sum_k a_k i_k^2\tau_k^2\int_0^\infty|\phi(j\omega)|^2d\omega \qquad \dots \quad (3.9)$$

Suppose now that the network whose noise is being observed is a circuit whose transfer impedance is $Z(\omega)$, then

$$\phi(j\omega)=Z(\omega)$$

Thus the mean-square-voltage-fluctuation is given by

$$\overline{e^2}=\frac{1}{\pi}\sum_k a_k i_k^2\tau_k^2\int_0^\infty[Z(\omega)]^2d\omega \qquad \dots \quad (3.10)$$

Here a deviation will be made from Campbell and Francis treatment and the value of $\Sigma a_k i_k^2\tau_k^2$ will be calculated by applying equation (1.1) which holds only for a resistor in thermal equilibrium. This would make the calculation more simple and straightforward. We note that

$i_k\tau_k$ = charge conveyed in a single event in the kth generator

and $i_k^2\tau_k^2=\epsilon_k^2$ = square-charge conveyed in a single event in the kth generator and $\Sigma a_k\tau i_k^2\tau_k^2$ = mean-square-charge conveyed by all the generators over a time τ.

Hence

$$\Sigma a_k i_k^2\tau_k^2=\frac{\overline{Q^2}}{\tau}=\frac{2kT}{R}\ \text{from (1.1)} \qquad \dots \quad (3.11)$$

Thus we get from (3.10)

$$\overline{e^2}=\frac{1}{\pi}\frac{2kT}{R}\int_0^\infty|Z(\omega)|^2d\omega$$

which is identical with (1.2)

UNIFIED THEORY: FÜRTH

Fürth has given a unified theory from thermodynamical reasonings. The main advantage of this treatment, as compared to that of Campbell and Francis, is that it deals with macroscopic quantities only and in the final expression for noise only such quantities appear. However, Fürth had anticipated several objections against the application of thermodynamic reasonings to conditions as exist in a valve and had tried to answer them.

But, as shown in the discussion, the objections have not been fully met with. To understand these discussions the arguments that have been advanced against the thermodynamical treatment and Fürth's answers to them are given below.

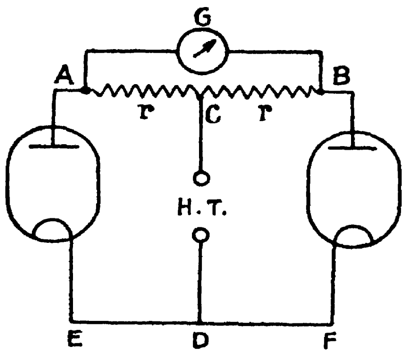
(*a*) Whereas, shot noise is associated with the flow of an average current i_s, thermal noise exists even in its absence.

(*b*) There is no temperature equilibrium in the case of a valve and the velocity distribution law is essentially asymmetrical.

(*c*) The external circuit is at different temperature from that of a valve.

Fürth's answers to these objections are as follows :

(*a*) Thermal noise in a conductor remains unaltered even when a non-vanishing steady current is flowing. Also shot noise exists even when no mean current flows. Now, consider the circuit shown in figure 3 (*a*) in



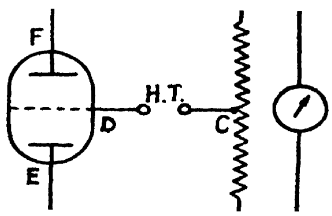


FIG. 3

which the two valves are exactly similar. In this case no mean current flows through the galvanometer G. Yet this will certainly register independent shot fluctuations due to the two halves, $ACDEA$ and $BCDFB$.

(*b*) Figures 3 (*a*) and 3 (*b*) are equivalent and Fürth contends that the two symmetrically placed emitting electrodes would render the velocity distribution law symmetrical.

(*c*) As long as $2rT << \frac{dV}{di}\ T$, thermal fluctuations in the external circuit would remain small enough to be negligible and in such a case the third point of objection can be waived,

Fürth then assumes that the noise currents are produced by an irregular veltage fluctuation δV_a originating in the inter-electrode space and another similar quantity δV_c originating in the surface layer of the cathode. On this basis the mean square-fluctuation current for the double-cathode valve shewn in figure 3 (*b*) is found to be

$$\overline{\delta i^2} = 4kT\left(g_c \frac{\delta V_c}{\delta V} + g_a \frac{\delta V_a}{\delta V}\right) df$$

where

$$g_a = \left(\frac{\overline{\delta i}}{\delta V}\right)_{i_s}, \quad g_c = \left(\frac{\overline{\delta i}}{\delta i_s}\right)\frac{e i_s}{kT}$$

bar indicating the average values. Fürth then proceeds with the argument that the fluctuations observed across $2r$ due to the double-cathode valve should be just twice that due to a single similar cathode in an asymmetric case. Thus for a single-cathode valve

$$\overline{\delta i^2} = 2kT\left(g_c \frac{\delta V_c}{\delta v} + g_a \frac{\delta V_a}{\delta V}\right) df \qquad \ldots \quad (4.1)$$

This expression has been used by Fürth to obtain expression for noise of a valve carrying (*i*) temperature-limited current, (*ii*) space-charge-limited current and (*iii*) retarding field current.

DISCUSSIONS

(*A*) *Campbell and Francis' theory.*—The success of the treatment of Campbell and Francis in explaining the different types of noise proves conclusively the basic identity between valve and circuit noise. Its treatment of both the phenomena is based on statistical reasonings and, as such, helps to bring out clearly the fact that noise is of microscopic origin. The method of mathematical treatment, though a little involved, is quite elegant. Perhaps the only drawback of the treatment is that it is not applicable to a non-linear device. But, as shown by the author this is not an insuperable difficulty. Subject to certain approximations one can extend the treatment to the case of a non-linear device, like a valve under condition of space-charge-limitation. This extension, incidentally, leads to an interpretation of the phenomenon of space-charge-reduction of noise, from a new angle, which, though qualitative, is more convincing and straightforward than the earlier theories. For example, the earlier theories sought to attribute the reduction to the increase in the magnitude of off-cathode potential dip E_m with the increase in space-current-density. But, this would mean that the chance of an electron at any instant, being thrown into the inter-electrode space and passed to the anode, depends on the space-current-density and hence on the electrons emitted at the preceding instants, that is, on some amount of correlation in the elementary events giving rise to noise. But, as is well known, this is not true for space-charge-reduced shot noise. This difficulty is not encountered in the interpretation of the author.

(*B*) *Fürth's theory.*—As mentioned earlier the objections that have been raised against Fürth's theory are not fully met by the arguments advanced by him. In what follows we will first discuss the several defects in Fürth's arguments, as pointed out by McDonald (1948) and as also are apparent according to the author of this paper. It will then be shown that the ultimate ground of all these objections against Fürth's theory is the lack of thermal equilibrium in a valve.

According to McDonald, the magnitude of the fluctuation current remains unaffected by the flow of a non-vanishing steady current in a metallic conductor only so long as the drift velocity thus produced is comparatively small—a condition which is wanting in a valve. Even in a metallic conductor, substantial deviation is likely to occur when the mean-free-path becomes very large. Again, the particular circuit arrangement in figure 3(*b*), proposed by Fürth (to show that the flow of a mean current is not necessary for the production of valve noise) is open to criticism because, although the detecting galvanometer G does not carry any mean current, the valves, which are the primary seats of noise, do in fact, carry mean currents. McDonald also points out that thermodynamical reasoning cannot be applied to the double-cathode valve of figure 3(*b*), because the two electron-streams inside this valve would present a symmetrical velocity-distribution only in the absence of the intervening grid. In the presence of this grid they are simply two asymmetrical beams inside a common enclosure.

Apart from the objections of McDonald, as discussed above, a close scrutiny shows that the theory has other serious difficulties. For example, one finds it difficult to agree with the suggestion that the total noise appearing across $2r$ in figure 3(*b*) would be just twice that of a single valve in the asymmetrical case. This is because an electron, which succeeds in penetrating the space-charge barrier in front of a cathode and is not captured by the grid, would introduce a partition type of noise which is absent in an ordinary diode. Further, it is not clear how the presence of the battery could be ignored in figure 3 (*b*).

All these objections, however, are essentially inter-connected and arise out of the one and the same factor, *viz.* the presence or the absence of thermal equilibrium. To show this we will analyse the objections to Fürth's arguments discussed above.

According to McDonald, the flow of mean-current is a differentiating factor between shot and thermal noise. Now, a valve is in effect a combination of an electronic switch and a resistance R_a in series. The current i_s is brought about in the process of the switching-on operation. Here the noise and the mean-current are two inevitable companions but are not inherently interdependent. It may also be noted that Schottky's expression gives the noise in a valve correctly, even if no mean current is flowing in the valve. This would remain valid even if a mean-current flows, provided thermal equilibrium is not disturbed. In a valve, however, due to the combined influence of large mean-free-path and accelerating field, the flow of mean-current is brought about through a process which disturbs the thermal equilibrium and that is why Nyquist's treatment is not applicable to it.

Let us next examine closely the arguments that a large mean-free-path would make a difference between the two types of noises. Let us consider, for example, a valve of the type shown in figure 3(b) from which the intervening grid has been removed. As will be shewn below, Nyquist's formula, which is applicable to thermal noise, is also applicable to it under certain conditions. We assume that the conditions are such that the mean-free-path of an electron is l, the separating distance between the two electrodes. Suppose also that the system is in thermal equilibrium. Under this condition cathode F will send a current i_0 (say) towards E which again would send an identical current towards F. The total noise in the valve would be twice the noise associated with these two currents and from Schottky's equation,

$$\delta i^2 = 4 i_0 \epsilon \, df. \qquad \ldots \quad (5.1)$$

If the electrons move through the same distance between two successive collisions and have all the same average thermal agitational velocity V and if the collisions are all mutually uncorrelated, then from simple application of electron theory the value of the resistance of the system is given by

$$R_a = \frac{6kTl}{nA\epsilon^2\lambda V}, \qquad \ldots \quad (5.2)$$

where A is the area of the cathodes and n the electronic concentration within the inter-electrode space.

For the case under consideration $\lambda = l$ and $nA/3$ represents the number of electrons crossing per unit area of a surface parallel to F and E per second. Half of this constitutes the number travelling in the same direction. Thus,

$$\frac{nA\epsilon V}{6} = i_0$$

From (5.1) we obtain

$$R_a = \frac{kT}{i_0 \epsilon} \qquad \ldots \quad (5.3)$$

From (5.1) and (5 3) we get

$$\overline{\delta i^2} = 4 \frac{kT}{R_a} df$$

which is identical with Nyquist's formula. It is thus obvious that shot and thermal noises would merge with one another only when thermal equilibrium exists, irrespective of the question of mean-free-path. A large drift velocity appears when large mean-free-path and an external voltage exist simultaneously and it affects the situation by rendering the velocity distribution asymmetrical i.e., by disturbing the thermal equilibrium.

It should be noted that relation (5,3) is not peculiar to the special case considered here. Meltzer (1949) has shown in a very simple manner, that it holds for any resistance in thermal equilibrium,

CONCLUDING REMARKS

It appears that thermal and valve noises have their origin in the same basic physical phenomenon, *viz.*, the existence of discrete carriers of electricity endowed with irregular agitational motion. This motion may or may not be in thermal equilibrium. For the resistor there is thermal equilibrium in agitational motion. For the case of the valve this is not so. A resistor and a valve differ from one another from the standpoint of noise in so far as thermal equilibrium may be assumed to exist in the case of the latter but not in the former. Hence one may look upon ordinary thermal noise as shot noise for the special case when thermal equilibrium exists and a valve may be looked upon as a special type of conductor devoid of thermal equilibrium. It, therefore, appears that the unified thermodynamical theory, as postulated by Fürth for ordinary valves, is improbable. Compared to this, the theory proposed by Campbell and Francis is distinctly a better approach to the problem. It is the elementary electrons which give rise to noise and the fact that our ultimate observations are concerned with macroscopic quantities only does not, in any way, affect the position. Hence a statistical treatment of the problem is more rigourous and elegant.

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INSTITUTE OF RADIO PHYSICS AND ELECTRONICS,
UNIVERSITY OF CALCUTTA

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